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AN EFFICIENT POWER-TRANSFORMED RATIO-TYPE ESTIMATOR OF FINITE POPULATION MEAN UNDER SIMPLE RANDOM SAMPLING

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ABSTRACT

This paper proposes an improved power-transformed ratio-type estimator for estimating the finite population mean under simple random sampling without replacement. The proposed estimator incorporates a power transformation together with the coefficient of variation, correlation coefficient, and sample size information through an optimally determined scalar parameter aimed at minimizing the mean squared error (MSE). Although the estimator may be viewed within the generalized framework of ratio-type estimators, it provides additional theoretical and empirical contributions through higher-order asymptotic investigations. Theoretical properties of the estimator, including bias and MSE, are derived up to the first and second orders of approximation using Taylor series expansion. The optimum value of the scalar parameter is obtained analytically, and conditions for improved efficiency over existing estimators are established. The performance of the proposed estimator is evaluated through extensive Monte Carlo simulation studies under normal, skewed, and heavy-tailed population distributions and is further validated using real-life datasets. The results demonstrate that the proposed estimator consistently yields lower MSE and higher percent relative efficiency (PRE) compared to classical and several modified ratio estimators, particularly in the presence of moderate to high correlation between the study and auxiliary variables. The efficiency gains remain evident across different population structures, indicating the robustness of the proposed estimator under departures from normality. The findings suggest that the proposed estimator is a reliable and efficient alternative for practical survey sampling applications.

Keywords: Auxiliary variable, Ratio estimator, Power transformation, Mean squared error, Efficiency.

1. Introduction

Accurately estimating population parameters from sample data is a fundamental problem in survey methodology. The population mean is a key parameter in survey methodology, frequently estimated for applications spanning economics, agriculture, health, environmental monitoring, and social sciences.

Because complete enumeration of a population is often infeasible due to budget, time, or logistical constraints, sampling methods are employed, and auxiliary information, data on a variable correlated with the study variable is frequently exploited to improve estimation precision.

The ratio estimator, introduced by Cochran (1940), remains one of the most widely used tools for incorporating auxiliary information. When the study variable Y and the auxiliary variable X are positively correlated and the regression line of Y on X is approximately linear through the origin, the ratio estimator achieves substantial gains over the simple random sample mean. Conversely, when negative correlation dominates, the product estimator of Murthy (1964) is preferred, while the regression estimator is adopted when the regression line of y on x has an intercept. A fundamental limitation of the classical ratio estimator is its reliance on the assumption that the regression line passes through the origin, a condition rarely satisfied in practice (Kadilar and Cingi, 2004; Singh and Tailor, 2003). This misspecification induces bias and reduces estimation efficiency. Consequently, a large body of literature has sought to modify the ratio estimator. Notable contributions include Sisodia and Dwivedi (1981), who incorporated the coefficient of variation of the auxiliary variable; Upadhyaya and Singh (1999), who used the coefficient of kurtosis; Singh and Tailor (2003), who exploited the known correlation coefficient; and Jerajuddin and Kishun (2016), who introduced sample size as a correction factor. More recently, Yadav *et al.* (2019), Muhammad *et al.* (2023), Yadav *et al*. (2024), Zaman *et al*. (2024), and Ekpenyong *et al.* (2025) have extended this literature further.

Despite the extensive literature on the ratio-type estimators, most existing studies rely on first- order approximations and evaluations under ideal distributional assumptions. As a result, the higher-order properties of power-transformed estimators and their performance under non-normal populations, such as skewed and heavy-tailed distributions, remain insufficiently explored. Since real survey data often deviates from normality, a more rigorous theoretical and empirical investigation is necessary to assess estimator performance under realistic conditions.

Motivated by this gap, this study proposes an efficient power-transformed ratio-type estimator of the population mean under simple random sampling without replacement (SRSWOR) by integrating the coefficient of variation, correlation coefficient, and sample size information through an optimally determined scaler parameter. Although the proposed estimator belongs to the generalized framework of Khoshnevisan *et al.* (2007), its contribution lies in the derivation of second-order bias and mean square error expressions, the establishment of efficiency conditions, and the assessment of robustness under normal, skewed, and heavy-tailed populations through Monte Carlo simulations and real-life datasets.

2. Methodology

2.1 Sampling Procedure, Notations and Review of Existing Estimators

Let $\zeta = \{\zeta_1, \zeta_2, ..., \zeta_N\}$ be a finite population of size *N* and let (y_τ, x_τ) be the value of the study variable Y and the auxiliary variable X on τ^{th} unit $\zeta_\tau, \tau = 1, 2, .., N$. Let $\bar{Y}$ and $\bar{X}$ be population means of the study variable Y and the auxiliary variable X respectively. Let a sample of size (n) be drawn by simple random sampling without replacement (SRSWOR) based on which the means ($\bar{x}$) and ($\bar{y}$) for the auxiliary variable (X) and the study variable (Y) are obtained. Thus, the following notations are defined

$\bar{y} = n^{-1}\sum_{\tau=1}^{n} y_\tau$: the sample mean of the study variable, $\bar{x} = n^{-1}\sum_{\tau=1}^{n} x_\tau$: the auxiliary variable sample mean, $\bar{Y} = N^{-1}\sum_{\tau=1}^{N} Y_\tau$: the study variable population mean, $\bar{X} = N^{-1}$: the auxiliary variable population mean, $C_x = S_x \bar{X}^{-1}$ **:** the auxiliary variable coefficients of variation, $C_y = S_y \bar{Y}^{-1}$: the study variable coefficients of variation, $\rho = S_{xy}\left(S_y S_x\right)^{-1}$: the correlation coefficient between the auxiliary and the study variables $S_x^2 = \left(N-1\right)^{-1}\sum_{\tau=1}^{N}\left(X_\tau - \bar{X}\right)^2$: the auxiliary variable population variance, $S_y^2 = \left(N-1\right)^{-1}\sum_{\tau=1}^{N}\left(Y_\tau - \bar{Y}\right)^2$: the study variable population variance, $S_{xy} = \left(N-1\right)^{-1}\sum_{\tau=1}^{N}\left(X_\tau - \bar{X}\right)\left(Y_\tau - \bar{Y}\right)$: the covariance between the study and auxiliary variables, $\beta_1 = \left[\left(N-1\right)\left(N-2\right)S_x^2\right]^{-1} N\sum_{\tau=1}^{N}\left(X_\tau - \bar{X}\right)^3$: the auxiliary variable coefficient of skewness, $\beta_2 = \left[\left(N-1\right)\left(N-2\right)\left(N-3\right)S_x^4\right]^{-1} N\left(N+1\right)\sum_{\tau=1}^{N}\left(X_\tau - \bar{X}\right)^4 - \left[\left(N-2\right)\left(N-3\right)\right]^{-1} 3\left(N-1\right)^2$: the auxiliary variable coefficient of kurtosis, M_x : the auxiliary variable median.

The population mean estimator along with its variance are given as

$$t_0 = \frac{\sum_{\tau=1}^{n} y_\tau}{n} \tag{1}$$

$$Var\left(t_0\right) = \bar{Y}^2 \eta C_y^2 \tag{2}$$

Cochran (1940) suggested the classical ratio estimator for estimating population mean ($\bar{Y}$) of the study variable (Y) by the use of auxiliary information (X). The suggested estimator, along with its bias and mean square error is given, respectively as

$$t_r = \frac{\bar{y}}{\bar{x}}\bar{X} \tag{3}$$

$$Bias(t_r) = \bar{Y}\eta\left[C_x^2 - \rho C_y C_x\right] \tag{4}$$

$$\mathrm{MSE}(t_r) = \bar{Y}^2\eta\left[C_y^2 + C_x^2 - 2\rho C_y C_x\right] \tag{5}$$

Sisodia and Dwivedi (1981) considered the use of known value of coefficient of variation (C_x) of the auxiliary variable and proposed ratio type estimator of the population mean. The proposed estimator, along with its bias and mean square error is given, respectively as

$$t_1 = \bar{y}\left(\frac{\bar{X} + C_x}{\bar{x} + C_x}\right) \tag{6}$$

$$Bias(t_1) = \bar{Y}\eta\left[\varphi_1^2 C_x^2 - \varphi_1 \rho C_y C_x\right] \tag{7}$$

$$\mathrm{MSE}(t_1) = \bar{Y}^2\eta\left[C_y^2 + \varphi_1^2 C_x^2 - 2\varphi_1 \rho C_y C_x\right] \tag{8}$$

where $\varphi_1 = \frac{\bar{X}}{\bar{X} + C_x}$

Building on the work of Sisodia and Dwivedi (1981), Upadhyaya and Singh (1999) proposed a ratio-type estimator for the population mean by incorporating the coefficient of variation (C_x) and coefficient of kurtosis (β_2) of the auxiliary variable. The proposed estimator, along with its bias and mean square error, is given as follows

$$t_2 = \bar{y}\left(\frac{\bar{X}C_x + \beta_2}{\bar{x}C_x + \beta_2}\right) \tag{9}$$

$$Bias(t_2) = \bar{Y}\eta\left[\varphi_2^2 C_x^2 - \varphi_2 \rho C_y C_x\right] \tag{10}$$

$$\text{MSE}(t_2) = \bar{Y}^2\eta\left[C_y^2 + \varphi_2^2 C_x^2 - 2\varphi_2 \rho C_y C_x\right] \tag{11}$$

where $\varphi_2 = \dfrac{\bar{X}C_x}{\bar{X}C_x + \beta_2}$

Extending Cochran's (1940) work, Singh and Tailor (2003) proposed a new estimator for the population mean by incorporating the coefficient of correlation (ρ) of the auxiliary variable. The estimator's expression, bias, and mean square error are presented below.

$$t_3 = \bar{y}\left(\frac{\bar{X} + \rho}{\bar{x} + \rho}\right) \tag{12}$$

$$Bias(t_3) = \bar{Y}\eta\left[\varphi_3^2 C_x^2 - \varphi_3 \rho C_y C_x\right] \tag{13}$$

$$\text{MSE}(t_3) = \bar{Y}^2\eta\left[C_y^2 + \varphi_3^2 C_x^2 - 2\varphi_3 \rho C_y C_x\right] \tag{14}$$

where $\varphi_3 = \dfrac{\bar{X}}{\bar{X} + \rho}$

Singh *et al.* (2004) considered the work Singh and Tailor (2003) and proposed a new ratio estimator of population mean by incorporating coefficient of kurtosis (β_2) of the auxiliary variable. The proposed estimator, along with its bias and mean square error is presented below.

$$t_4 = \bar{y}\left(\frac{\bar{X} + \beta_2}{\bar{x} + \beta_2}\right) \tag{15}$$

$$Bias(t_4) = \bar{Y}\eta\left[\varphi_4^2 C_x^2 - \varphi_4 \rho C_y C_x\right] \tag{16}$$

$$\text{MSE}(t_4) = \bar{Y}^2\eta\left[C_y^2 + \varphi_4^2 C_x^2 - 2\varphi_4 \rho C_y C_x\right] \tag{17}$$

where $\varphi_4 = \dfrac{\bar{X}}{\bar{X} + \beta_2}$

Yan and Tian (2010) defined a new ratio type estimator of population mean by incorporating known value of coefficient of skewness (β_1) of the auxiliary variable. The proposed estimator, along with its bias and mean square error, is given below.

$$t_5 = \bar{y}\left(\frac{\bar{X} + \beta_1}{\bar{x} + \beta_1}\right) \tag{18}$$

$$Bias(t_5) = \bar{Y}\eta\left[\varphi_5^2 C_x^2 - \varphi_5 \rho C_y C_x\right] \tag{19}$$

$$\text{MSE}(t_5) = \bar{Y}^2\eta\left[C_y^2 + \varphi_5^2 C_x^2 - 2\varphi_5 \rho C_y C_x\right] \tag{20}$$

where $\varphi_5 = \dfrac{\bar{X}}{\bar{X} + \beta_1}$

Subramani and Kumarpandiyan (2013) developed a new ratio type estimator of population mean by incorporating known value of median (M_x) of the auxiliary variable. The proposed estimator, along with its bias and mean square error, is presented below.

$$t_6 = \bar{y}\left(\frac{\bar{X}+M_x}{\bar{x}+M_x}\right) \tag{21}$$

$$Bias(t_6) = \bar{Y}\eta\left[\varphi_6^2 C_x^2 - \varphi_6 \rho C_y C_x\right] \tag{22}$$

$$\text{MSE}(t_6) = \bar{Y}^2\eta\left[C_y^2 + \varphi_6^2 C_x^2 - 2\varphi_6 \rho C_y C_x\right] \tag{23}$$

where $\varphi_6 = \dfrac{\bar{X}}{\bar{X}+M_x}$

Jerajuddin and Kishun (2016) in order to enhance the precision of Cochran (1940) estimator of population mean, introduced ample size of the sample of population. The formula for the estimator, together with its bias and mean square error, appears below.

$$t_7 = \bar{y}\left(\frac{\bar{X}+n}{\bar{x}+n}\right) \tag{24}$$

$$Bias(t_7) = \bar{Y}\eta\left[\varphi_7^2 C_x^2 - \varphi_7 \rho C_y C_x\right] \tag{25}$$

$$\text{MSE}(t_7) = \bar{Y}^2\eta\left[C_y^2 + \varphi_7^2 C_x^2 - 2\varphi_7 \rho C_y C_x\right] \tag{26}$$

where $\varphi_7 = \dfrac{\bar{X}}{\bar{X}+n}$

Yadav *et al.* (2019) modified the estimator proposed by Jerajuddin and Kishun (2016) by incorporating coefficient of correlation (ρ). Below are the proposed estimator, its bias, and its mean square error.

$$t_8 = \bar{y}\left(\frac{\bar{X}\rho+n}{\bar{x}\rho+n}\right) \tag{27}$$

$$Bias(t_8) = \bar{Y}\eta\left[\varphi_8^2 C_x^2 - \varphi_8 \rho C_y C_x\right] \tag{28}$$

$$\text{MSE}(t_8) = \bar{Y}^2\eta\left[C_y^2 + \varphi_8^2 C_x^2 - 2\varphi_8 \rho C_y C_x\right] \tag{29}$$

where $\varphi_8 = \dfrac{\bar{X}\rho}{\bar{X}\rho+n}$

Zakari *et al.* (2020) considered the use of unknown weight and defined a new ratio type estimator of population mean. The proposed estimator, along with its bias and mean square error respectively is given as

$$t_9 = \bar{y}\alpha\left(\frac{\bar{X}+n}{\bar{x}+n}\right) \tag{30}$$

$$Bias(t_9) = \bar{Y}\left[(\alpha-1) + \eta\varphi_9^2 C_x^2 - \eta\varphi_9 \rho C_y C_x\right] \tag{31}$$

$$MSE(t_9) = \bar{Y}^2\left[1 - \frac{\left(1+\eta\varphi_9^2 C_x^2 - \eta\varphi_9 \rho C_y C_x\right)^2}{1+\eta C_y^2 + 3\eta\varphi_9^2 C_x^2 - 4\eta\varphi_9 \rho C_y C_x}\right] \tag{32}$$

where $\varphi_9 = \dfrac{\bar{X}}{\bar{X}+n}$ and $\alpha = \dfrac{1+\eta\varphi_9^2 C_x^2 - \eta\varphi_9 \rho C_y C_x}{1+\eta C_y^2 + 3\eta\varphi_9^2 C_x^2 - 4\eta\varphi_9 \rho C_y C_x}$

2.2 Proposed Estimator

In this section, a new ratio type estimator of the population mean is proposed by incorporating known value of the coefficient of correlation of the auxiliary variable in order to improve on the precisions of Jerajuddin and Kishun (2016) and Yadav *et al.* (2019) ratio type estimators. The proposed estimator is given as

$$t^{NT} = \bar{y}\left(\frac{n\bar{X}\rho + C_x}{n\bar{x}\rho + C_x}\right)^{\gamma} \tag{33}$$

where γ is a suitable chosen unknown weight or constant to be determined such that the mean square error of the proposed estimator is minimum.

The asymptotic properties of the proposed estimator are obtained by defining the following terms

$$\text{Let } \zeta_y = \frac{\bar{y}-\bar{Y}}{\bar{Y}} \text{ and } \zeta_x = \frac{\bar{x}-\bar{X}}{\bar{X}}, \text{ such that } \bar{y} = \bar{Y}\left[1+\zeta_y\right] \text{ and } \bar{x} = \bar{X}\left[1+\zeta_x\right]$$

$$\left.\begin{aligned} &\mathrm{E}\left(\zeta_y\right) = \mathrm{E}\left(\zeta_x\right) = 0 \\ &\mathrm{E}\left(\zeta_y^2\right) = \eta C_y^2, \mathrm{E}\left(\zeta_x^2\right) = \eta C_x^2, \ \mathrm{E}\left(\zeta_x\zeta_y\right) = \eta\rho C_y C_x \end{aligned}\right\} \tag{34}$$

where $\eta = \left(\frac{1}{n} - \frac{1}{N}\right) = \frac{1-f}{n}$. Expressing (33) in terms of ζ_y and ζ_x gives

$$t^{NT} = \bar{Y}\left[1+\zeta_y\right]\left[\frac{n\bar{X}\rho + C_x}{n\bar{X}\rho\left(1+\zeta_x\right) + C_x}\right]^{\gamma} \tag{35}$$

Factorizing the second component of the RHS of (35) gives

$$t^{NT} = \bar{Y}\left[1+\zeta_y\right]\left[\frac{n\bar{X}\rho + C_x}{n\bar{X}\rho + C_x\left(1+\frac{n\bar{X}\rho\zeta_x}{n\bar{X}\rho + C_x}\right)}\right]^{\gamma} = \bar{Y}\left[1+\zeta_y\right]\left[1+\varphi\zeta_x\right]^{-\gamma} \tag{36}$$

where $\varphi = \frac{n\bar{X}\rho}{n\bar{X}\rho + C_x}$

If it is assumed that $\left|\varphi\zeta_x\right| < 1$, then the expression $\left[1+\varphi\zeta_x\right]^{-\gamma}$ can be expanded to convergent infinite series using binomial expansion, and approximating to the first and second orders.

By first order of approximation, terms of ζ's greater than two (2) are neglected, and thus, (36) gives

$$t^{NT} = \bar{Y}\left[1+\zeta_y\right]\left[1-\gamma\varphi\zeta_x + \frac{\gamma\left(\gamma+1\right)}{2}\varphi^2\zeta_x^2\right] \tag{37}$$

Multiplying out the brackets gives

$$t^{NT} = \bar{Y} + \bar{Y}\left[\zeta_y - \gamma\varphi\zeta_x + \frac{\gamma\left(\gamma+1\right)}{2}\varphi^2\zeta_x^2 - \gamma\varphi\zeta_x\zeta_y\right] \tag{38}$$

Subtracting $\bar{Y}$ and taking expectation to both sides of (38)

$$\mathrm{E}\left[t^{NT} - \bar{Y}\right] = \mathrm{E}\left[\bar{Y}\left(\zeta_y - \gamma\varphi\zeta_x + \frac{\gamma\left(\gamma+1\right)}{2}\varphi^2\zeta_x^2 - \gamma\varphi\zeta_x\zeta_y\right)\right] \tag{39}$$

Applying results of (34), then, the bias of the estimator t^{NT} is obtained as

$$Bias\left(t^{NT}\right) = \bar{Y}\eta\left[\frac{\gamma\left(\gamma+1\right)}{2}\varphi^2 C_x^2 - \gamma\varphi\rho C_y C_x\right] \tag{40}$$

Also, subtracting, taking expectation and squaring both sides of (38) gives

$$\mathrm{E}\left[t^{NT}-\bar{Y}\right]^2=\mathrm{E}\left[\bar{Y}\left(\zeta_y-\gamma\varphi\zeta_x+\frac{\gamma(\gamma+1)}{2}\varphi^2\zeta_x^2-\gamma\varphi\zeta_x\zeta_y\right)\right]^2 \tag{41}$$

Applying results of (34), then, the mean square error (MSE) of the estimator t^{NT} is obtained as

$$MSE\left(t^{NT}\right)=\bar{Y}^2\eta\left[C_y^2+\gamma^2\varphi^2C_x^2-2\gamma\varphi\rho C_yC_x\right] \tag{42}$$

To obtain the optimum value of γ, we differentiate (42) partially with respect to γ and equate to zero as

$$\frac{\partial MSE\left(t^{NT}\right)}{\partial\gamma}=\bar{Y}^2\eta\left[2\gamma\varphi^2C_x^2-2\varphi\rho C_yC_x\right]=0$$

$$2\gamma\varphi^2C_x^2=2\varphi\rho C_yC_x$$

$$\gamma_{opt}=\frac{\rho C_y}{\varphi C_x} \tag{43}$$

By substituting the value of (43) in (42), the minimum mean square error is obtained as

$$MSE\left(t^{NT}\right)_{\min}=\bar{Y}^2\eta\left[C_y^2+\left(\frac{\rho C_y}{\varphi C_x}\right)^2\varphi^2C_x^2-2\left(\frac{\rho C_y}{\varphi C_x}\right)\varphi\rho C_yC_x\right] \tag{44}$$

Simplifying (44) further, the minimum mean square error is obtained as

$$MSE\left(t^{NT}\right)_{\min}=\bar{Y}^2\eta C_y^2\left[1-\rho^2\right] \tag{45}$$

Approximation to second order, terms of $\zeta's$ greater than four (4) are neglected, and thus, (36) yields

$$t^{NT}=\bar{Y}\left[1+\zeta_y\right]\left[1-\gamma\varphi\zeta_x+\frac{\gamma(\gamma+1)}{2}\varphi^2\zeta_x^2-\frac{\gamma(\gamma+1)(\gamma+2)}{6}\varphi^3\zeta_x^3+\frac{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}{24}\varphi^4\zeta_x^4\right] \tag{46}$$

Let: $\psi_1=\gamma\varphi$, $\psi_2=\frac{\gamma(\gamma+1)}{2}\varphi^2$, $\psi_3=\frac{\gamma(\gamma+1)(\gamma+2)}{6}\varphi^3$, and $\psi_4=\frac{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}{24}\varphi^4$

$$\Rightarrow\quad t^{NT}=\bar{Y}\left[1+\zeta_y\right]\left[1-\psi_1\zeta_x+\psi_2\zeta_x^2-\psi_3\zeta_x^3+\psi_4\zeta_x^4\right] \tag{47}$$

Expanding and grouping by order yields

$$t^{NT}=\bar{Y}+\bar{Y}\left[\left(\zeta_y-\psi_1\zeta_x\right)+\left(\psi_2\zeta_x^2-\psi_1\zeta_y\zeta_x\right)+\left(-\psi_3\zeta_x^3+\psi_2\zeta_y\zeta_x^2\right)+\left(\psi_4\zeta_x^4-\psi_3\zeta_y\zeta_x^3\right)\right] \tag{48}$$

Subtracting $\bar{Y}$ and from both sides of (48)

$$t^{NT}-\bar{Y}=\bar{Y}\left[\left(\zeta_y-\psi_1\zeta_x\right)+\left(\psi_2\zeta_x^2-\psi_1\zeta_y\zeta_x\right)+\left(-\psi_3\zeta_x^3+\psi_2\zeta_y\zeta_x^2\right)+\left(\psi_4\zeta_x^4-\psi_3\zeta_y\zeta_x^3\right)\right] \tag{49}$$

$$t^{NT}-\bar{Y}=\bar{Y}\left[\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3+\mathrm{A}_4\right] \tag{50}$$

where $\mathrm{A}_1=\zeta_y-\gamma\varphi\zeta_x$, $\mathrm{A}_2=\frac{\gamma(\gamma+1)}{2}\varphi^2\zeta_x^2-\gamma\varphi\zeta_y\zeta_x$, $\mathrm{A}_3=-\frac{\gamma(\gamma+1)(\gamma+2)}{6}\varphi^3\zeta_x^3+\frac{\gamma(\gamma+1)}{2}\varphi^2\zeta_y\zeta_x^2$

and $\mathrm{A}_4=\frac{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}{24}\varphi^4\zeta_x^4-\frac{\gamma(\gamma+1)(\gamma+2)}{6}\varphi^3\zeta_y\zeta_x^3$

Hence, the bias of the second order of approximation of t^{NT} is obtained as

$$Bias\left(t^{NT}\right)=\mathrm{E}\left(t^{NT}-\bar{Y}\right)=\bar{Y}\left[\mathrm{E}\left(\mathrm{A}_1\right)+\mathrm{E}\left(\mathrm{A}_2\right)+\mathrm{E}\left(\mathrm{A}_3\right)+\mathrm{E}\left(\mathrm{A}_4\right)\right] \tag{51}$$

Thus,

$$\mathrm{E}\left(\mathrm{A}_1\right)=\mathrm{E}\left(\zeta_y-\gamma\varphi\zeta_x\right)=\mathrm{E}\left(\zeta_y\right)-\gamma\varphi\mathrm{E}\left(\zeta_x\right)=0 \tag{52}$$

$$\mathrm{E}\left(\mathrm{A}_2\right)=\frac{\gamma(\gamma+1)}{2}\varphi^2\mathrm{E}\left(\zeta_x^2\right)-\gamma\varphi\mathrm{E}\left(\zeta_y\zeta_x\right)=\frac{1-f}{n}\left[\frac{\gamma(\gamma+1)}{2}\varphi^2C_x^2-\gamma\varphi\rho C_yC_x\right] \tag{53}$$

$$\mathrm{E}(\mathrm{A}_3) = -\frac{\gamma(\gamma+1)(\gamma+2)}{6}\varphi^3\mathrm{E}\left(\zeta_x^3\right) + \frac{\gamma(\gamma+1)}{2}\varphi^2\mathrm{E}\left(\zeta_y\zeta_x^2\right)$$
$$= \frac{(1-f)(1-2f)}{n^2}.\frac{\gamma(\gamma+1)}{2}\varphi^2\left[\lambda_{2yx} - \frac{\varphi(\gamma+2)}{3}\lambda_{3x}\right] \quad (54)$$

where $\lambda_{2yx} = \frac{\mu_{2yx}}{\bar{Y}\bar{X}^2}$, $\lambda_{3x} = \frac{\mu_{3x}}{\bar{X}^3}$, $\mu_{3x} = (N-1)^{-1}\sum_{\tau=1}^{N}\left(X_\tau - \bar{X}\right)^3$ and $\mu_{2yx} = (N-1)^{-1}\sum_{\tau=1}^{N}\left(Y_\tau - \bar{Y}\right)\left(X_\tau - \bar{X}\right)^2$

$$\mathrm{E}(\mathrm{A}_4) = \frac{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}{24}\varphi^4\mathrm{E}\left(\zeta_x^4\right) - \frac{\gamma(\gamma+1)(\gamma+2)}{6}\varphi^3\mathrm{E}\left(\zeta_y\zeta_x^3\right)$$
$$= \frac{(1-f)^2}{n^2}.\frac{\gamma(\gamma+1)(\gamma+2)}{2}\varphi^3\left[\frac{\varphi(\gamma+3)}{4}C_x^4 - \rho C_y C_x^3\right] \quad (55)$$

Substituting (52), (53), (54) and (55) in (51) yields

$$Bias\left(t^{NT}\right) = \frac{1-f}{n}\bar{Y}\left[\frac{\gamma(\gamma+1)}{2}\varphi^2C_x^2 - \gamma\varphi\rho C_yC_x\right] + \frac{(1-f)(1-2f)}{n^2}\bar{Y}.\frac{\gamma(\gamma+1)}{2}\varphi^2\left[\lambda_{2yx} - \frac{\varphi(\gamma+2)}{3}\lambda_{3x}\right]$$
$$+ \frac{(1-f)^2}{n^2}\bar{Y}.\frac{\gamma(\gamma+1)(\gamma+2)}{2}\varphi^3\left[\frac{\varphi(\gamma+3)}{4}C_x^4 - \rho C_yC_x^3\right]$$
$$= \eta\bar{Y}\left\{\begin{array}{l}\left[\frac{\gamma(\gamma+1)}{2}\varphi^2C_x^2 - \gamma\varphi\rho C_yC_x\right] + \frac{(1-2f)}{n}.\frac{\gamma(\gamma+1)}{2}\varphi^2\left[\lambda_{2yx} - \frac{\varphi(\gamma+2)}{3}\lambda_{3x}\right]\\ +\eta.\frac{\gamma(\gamma+1)(\gamma+2)}{2}\varphi^3\left[\frac{\varphi(\gamma+3)}{4}C_x^4 - \rho C_yC_x^3\right]\end{array}\right\} \quad (56)$$

The mean square error of second order of approximation of t^{NT} is obtained by squaring and taking expectation on both sides of (50). That is,

$$MSE\left(t^{NT}\right) = \mathrm{E}\left(t^{NT} - \bar{Y}\right)^2 = \bar{Y}^2\mathrm{E}\left[\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{A}_4\right]^2 \quad (57)$$

But, $\mathrm{E}\left[\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{A}_4\right]^2 \simeq \mathrm{E}\left(\mathrm{A}_1^2\right) + 2\mathrm{E}\left(\mathrm{A}_1\mathrm{A}_2\right) + 2\mathrm{E}\left(\mathrm{A}_1\mathrm{A}_3\right) + \mathrm{E}\left(\mathrm{A}_2^2\right)$

$$\Rightarrow \quad \mathrm{E}\left(\mathrm{A}_1^2\right) = \mathrm{E}\left(\zeta_y - \gamma\varphi\zeta_x\right)^2 = \mathrm{E}\left(\zeta_y^2\right) - 2\gamma\varphi\mathrm{E}\left(\zeta_y\zeta_x\right) + \gamma^2\varphi^2\mathrm{E}\left(\zeta_x^2\right) = \frac{1-f}{n}\left[C_y^2 + \gamma^2\varphi^2C_x^2 - 2\gamma\varphi\rho C_yC_x\right] \quad (58)$$

$$2\mathrm{E}\left(\mathrm{A}_1\mathrm{A}_2\right) = 2\left[-\frac{\gamma^2(\gamma+1)}{2}\varphi^3\mathrm{E}\left(\zeta_x^3\right) + \frac{\gamma(3\gamma+1)}{2}\varphi^2\mathrm{E}\left(\zeta_y\zeta_x^2\right) - \gamma\varphi\mathrm{E}\left(\zeta_y^2\zeta_x\right)\right]$$
$$= \frac{2(1-f)(1-2f)}{n^2}\left[-\frac{\gamma^2(\gamma+1)}{2}\varphi^3\lambda_{3x} + \frac{\gamma(3\gamma+1)}{2}\varphi^2\lambda_{2yx} - \gamma\varphi\lambda_{y2x}\right] \quad (59)$$

where $\lambda_{y2x} = \frac{\mu_{y2x}}{\bar{Y}^2\bar{X}}$ and $\mu_{y2x} = (N-1)^{-1}\sum_{\tau=1}^{N}\left(Y_\tau - \bar{Y}\right)^2\left(X_\tau - \bar{X}\right)$

$$2\mathrm{E}\left(\mathrm{A}_1\mathrm{A}_3\right) = 2\left[\frac{\gamma^2(\gamma+1)(\gamma+2)}{6}\varphi^4\mathrm{E}\left(\zeta_x^4\right) + \frac{\gamma(\gamma+1)}{2}\varphi^2\mathrm{E}\left(\zeta_y^2\zeta_x^2\right) - \frac{\gamma(\gamma+1)(2\gamma+1)}{6}\varphi^3\mathrm{E}\left(\zeta_y\zeta_x^3\right)\right]$$
$$= \frac{(1-f)^2}{n^2}.\gamma(\gamma+1)\varphi^2\left[\gamma(\gamma+2)\varphi^2C_x^4 + C_y^2C_x^2\left(1+2\rho^2\right) - 2(2\gamma+1)\varphi\rho C_yC_x^3\right] \quad (60)$$

$$\mathrm{E}\left(\mathrm{A}_2^2\right) = \frac{\gamma^2(\gamma+1)^2}{4}\varphi^4\mathrm{E}\left(\zeta_x^4\right) - \gamma^2(\gamma+1)\varphi^3\mathrm{E}\left(\zeta_y\zeta_x^3\right) + \gamma^2\varphi^2\mathrm{E}\left(\zeta_2^2\zeta_x^2\right)$$
$$= \frac{(1-f)^2}{n^2}\left[\frac{3\gamma^2(\gamma+1)^2}{4}\varphi^4C_x^4 - 3\gamma^2(\gamma+1)\varphi^3\rho C_yC_x^3 + \gamma^2\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)\right] \quad (61)$$

Inserting (58), (59), (60) and (61) in (57) results to

$$MSE\left(t^{NT}\right)=\eta\bar{Y}^2\left\{\begin{array}{l}\left[C_y^2+\gamma^2\varphi^2C_x^2-2\gamma\varphi\rho C_yC_x\right]+\dfrac{2(1-2f)}{n}\left[-\dfrac{\gamma^2(\gamma+1)}{2}\varphi^3\lambda_{3x}+\dfrac{\gamma(3\gamma+1)}{2}\varphi^2\lambda_{2yx}-\gamma\varphi\lambda_{y2x}\right]\\+\eta\gamma(\gamma+1)\varphi^2\left[\gamma(\gamma+2)\varphi^2C_x^4+C_y^2C_x^2\left(1+2\rho^2\right)-2(2\gamma+1)\varphi\rho C_yC_x^3\right]\\+\eta\left[\dfrac{3\gamma^2(\gamma+1)^2}{4}\varphi^4C_x^4-3\gamma^2(\gamma+1)\varphi^3\rho C_yC_x^3+\gamma^2\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)\right]\end{array}\right\} \quad (62)$$

To obtain the optimum value of γ, let:

$A=\dfrac{1-f}{n}\bar{Y}^2$, $B=\dfrac{2(1-f)(1-2f)}{n^2}\bar{Y}^2$ and $C=\dfrac{(1-f)^2}{n^2}\bar{Y}^2$. Then, mean square error becomes

$$\begin{aligned}MSE\left(t^{NT}\right)&=A\left[C_y^2+\gamma^2\varphi^2C_x^2-2\gamma\varphi\rho C_yC_x\right]+B\left[-\frac{\gamma^2(\gamma+1)}{2}\varphi^3\lambda_{3x}+\frac{\gamma(3\gamma+1)}{2}\varphi^2\lambda_{2yx}-\gamma\varphi\lambda_{y2x}\right]\\&+C\gamma(\gamma+1)\varphi^2\left[\gamma(\gamma+2)\varphi^2C_x^4+C_y^2C_x^2\left(1+2\rho^2\right)-2(2\gamma+1)\varphi\rho C_yC_x^3\right]\\&+C\left[\frac{3\gamma^2(\gamma+1)^2}{4}\varphi^4C_x^4-3\gamma^2(\gamma+1)\varphi^3\rho C_yC_x^3+\gamma^2\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)\right]\end{aligned} \quad (63)$$

Expanding and grouping all terms results to

$$\begin{aligned}MSE\left(t^{NT}\right)&=AC_y^2+\left[-2A\varphi\rho C_yC_x+\frac{B}{2}\varphi^2\lambda_{2yx}-B\varphi\lambda_{2yx}+C\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)-2C\varphi^3\rho C_yC_x^3\right]\gamma\\&+\left[A\varphi^2C_x^2-\frac{B}{2}\varphi^3\lambda_{3x}+\frac{3B}{2}\varphi^2\lambda_{2yx}+\frac{11C}{4}\varphi^4C_x^4+2C\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)-9C\varphi^3\rho C_yC_x^3\right]\gamma^2\\&+\left[-\frac{B}{2}\varphi^3\lambda_{3x}+\frac{9C}{2}\varphi^4C_x^4-10C\varphi^3\rho C_yC_x^3\right]\gamma^3+\left[\frac{7C}{4}\varphi^4C_x^4-3C\varphi^3\rho C_yC_x^3\right]\gamma^4\end{aligned} \quad (64)$$

Now, differentiating $MSE\left(t^{NT}\right)$ partially with respect to γ gives

$$\begin{aligned}\frac{\partial MSE\left(t^{NT}\right)}{\partial\gamma}&=\left[-2A\varphi\rho C_yC_x+\frac{B}{2}\varphi^2\lambda_{2yx}-B\varphi\lambda_{2yx}+C\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)-2C\varphi^3\rho C_yC_x^3\right]\\&+2\left[A\varphi^2C_x^2-\frac{B}{2}\varphi^3\lambda_{3x}+\frac{3B}{2}\varphi^2\lambda_{2yx}+\frac{11C}{4}\varphi^4C_x^4+2C\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)-9C\varphi^3\rho C_yC_x^3\right]\gamma\\&+\left[-\frac{3B}{2}\varphi^3\lambda_{3x}+\frac{27C}{2}\varphi^4C_x^4-30C\varphi^3\rho C_yC_x^3\right]\gamma^2+\left[7C\varphi^4C_x^4-12C\varphi^3\rho C_yC_x^3\right]\gamma^3\end{aligned} \quad (65)$$

Setting $\dfrac{\partial MSE\left(t^{NT}\right)}{\partial\gamma}=0$, gives a cubic equation. But since B and C are small (they are of second- order), we can first, approximate by setting $B=0$ and $C=0$

$$\Rightarrow \quad -2A\varphi\rho C_yC_x+2A\varphi^2C_x^2\gamma=0$$

$$\gamma_{opt}^{(1)}=\frac{\rho C_y}{\varphi C_x} \quad (66)$$

The result obtained in (66) is the same first-order result as before in (43).

To find the second-order correction, denoted as, δ by including B and C, let

$$\gamma_{opt}=\gamma_{opt}^{(1)}+\delta \quad (67)$$

Here, Perturbation theory is applied, since B and C are small. Now, $\gamma=\gamma^{(1)}+\delta$ is substituted into the derivative equation by keeping only terms up to order O(B) and O(C). The derivative equation is

$$-2A\varphi\rho C_yC_x+2A\varphi^2C_x^2\gamma+terms\ with\ B\ and\ C$$

For $\gamma=\gamma^{(1)}$, the order, O(A) terms sum to zero. That is,

$$-2A\varphi\rho C_yC_x+2A\varphi^2C_x^2\gamma^{(1)}=0 \quad (68)$$

Thus, we need

$$2A\varphi^2C_x^2\delta+\left[\text{all } B \text{ and } C \text{ terms evaluated at } \gamma^{(1)}\right]\approx 0$$

$$\delta\approx-\frac{1}{2A\varphi^2C_x^2}\left[\text{all } B \text{ and } C \text{ terms at } \gamma^{(1)}\right] \tag{69}$$

Writing all the B and C terms from the derivatives in (65), gives

$$\delta=-\frac{1}{2A\varphi^2C_x^2}\begin{bmatrix}\frac{B}{2}\varphi^2\lambda_{2yx}-B\varphi\lambda_{2yx}+C\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)-2C\varphi^3\rho C_yC_x^3+\\ 2\gamma^{(1)}\left(-\frac{B}{2}\varphi^3\lambda_{3x}+\frac{3B}{2}\varphi^2\lambda_{2yx}+\frac{11C}{4}\varphi^4C_x^4+2C\varphi^2C_y^2C_x^2\left(1+2\rho^2\right)-9C\varphi^3\rho C_yC_x^3\right)\\ +3\left(\gamma^{(1)}\right)^2\left(-\frac{3B}{2}\varphi^3\lambda_{3x}+\frac{27C}{2}\varphi^4C_x^4-30C\varphi^3\rho C_yC_x^3\right)+\left(\gamma^{(1)}\right)^3\left(7C\varphi^4C_x^4-12C\varphi^3\rho C_yC_x^3\right)\end{bmatrix} \tag{70}$$

Note: In practice, this full correction is very complicated, as most researchers use simpler approximation. Hence, the approximate second- order optimal γ is

$$\gamma_{opt}\approx\frac{\rho C_y}{\varphi C_x} \tag{71}$$

This is because, the correction, δ is of order $O\left(n^{-1}\right)$, and for large samples $\left(n>30\right)$, δ is very small. Hence, the first order γ is usually good enough. However, the full correction is needed when sample size, n is small $\left(n<20\right)$, when the population has high skewness, and when extremely high precision is needed.

2.3 Theoretical Efficiency Comparisons

Here, the efficiency of the proposed estimator is compared with some of the aforementioned estimators in the literature. Hence, the conditions under which the proposed estimator is more efficient are given as

i. The proposed estimator t^{NT} is more efficient than the estimator of the sample mean, if and only if

$$MSE\left(t^{NT}\right)<Var\left(t_0\right) \tag{72}$$

$$\bar{Y}^2\eta C_y^2\left[1-\rho^2\right]<\bar{Y}^2\eta C_y^2$$

$$1-\rho^2<1 \tag{73}$$

ii. The proposed estimator t^{NT} is more efficient than the usual estimator defined by Cochran (1940) if and only if

$$MSE\left(t^{NT}\right)<MSE\left(t_r\right) \tag{74}$$

$$\bar{Y}^2\eta C_y^2\left[1-\rho^2\right]<\bar{Y}^2\eta\left[C_y^2+C_x^2-2\rho C_yC_x\right]$$

$$C_y^2\left[\left(1-\rho^2\right)-1\right]<C_x^2-2\rho C_yC_x \tag{75}$$

iii. The proposed estimator t^{NT} is more efficient than the other ratio type estimators aforementioned in the reviewed literature if and only if

$$MSE\left(t^{NT}\right)<MSE\left(t_i\right),\tau=1,2,...,9 \tag{76}$$

$$\bar{Y}^2\eta C_y^2\left[1-\rho^2\right]<\bar{Y}^2\eta\left[C_y^2+\varphi_\tau^2C_x^2-2\varphi_\tau\rho C_yC_x\right]$$

$$C_y^2\left[\left(1-\rho^2\right)-1\right]<\varphi_\tau^2C_x^2-2\varphi_\tau\rho C_yC_x \tag{77}$$

3. Results and Discussion

3.1 Simulation Study

To evaluate the performance of the proposed estimator, we conduct a simulation study using synthetic populations generated from the model proposed by Singh and Horn (1998), which is defined as:

$$\left.\begin{aligned} Y &= 10 + \sqrt{1-\rho^2}\,Y^* + \rho\left(\frac{S_y}{S_x}\right)X^* \\ X &= 5 + X^* \end{aligned}\right\} \tag{78}$$

where X^* and Y^* are independent random variables generated from base distributions: Normal (symmetric), Gamma (skewed) and t-distribution (heavy-tailed) with $\rho = 0.70, 0.80, 0.95$. The simulation enables comprehensive comparison of the proposed estimator in terms of precision and efficiency. Thus, the simulation algorithms comprise the following:

(i) Three finite populations are generated based on the model defined in (78).
   (a) Generate a population of size $N = 1000$ from normal distribution, $X^* \sim N(0,1)$ and $Y^* \sim N(0,1)$
   (b) Generate another population of size $N = 1000$ from gamma distribution, $X^* \sim Gamma(2,1)$ and $Y^* \sim Gamma(3,1)$
   (c) Similarly, generate population of size $N = 1000$ from t distribution, $X^* \sim t_{(4)}$ and $X^* \sim t_{(5)}$

(ii) From the generated populations, a simple random sample of sizes n=50,100, 150, and 200 are drawn without replacement. For each sample, the descriptive statistics are computed.

(iii) The process above is repeated for $B = 10000$ Monte Carlo replications. The performance of the estimators is measured using mean square error (MSE) with respect to the population mean of the study variable and percent relative efficiency (PRE), defined as

$$MSE(t) = \frac{1}{B}\sum_{\tau=1}^{B}\left(t_\tau - \bar{Y}\right)^2 \tag{79}$$

$$PRE = \frac{MSE(t_0)}{MSE(t)} \times 100 \tag{80}$$

where t represents the reference estimator.

3.1.1 Simulation Results

Table 1: Bias, MSEs and PREs of Estimators from the simulated data using $(n = 50, \rho = 0.7)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0023	0.0189	100.0000	0.0008	0.0200	100.0000	-0.0017	0.0183	100.0000
t_r	0.0034	0.0419	45.1184	0.0082	0.0399	50.0724	0.0043	0.0432	42.3187
t_1	0.0031	0.0382	49.4763	0.0076	0.0363	54.9987	0.0038	0.0394	46.4187
t_2	0.0012	0.0105	180.2843	0.0010	0.0129	155.0195	-0.0018	0.0118	155.2854
t_3	0.0025	0.0310	60.9948	0.0064	0.0291	68.7208	0.0028	0.0320	57.1820
t_4	0.0012	0.0156	121.4408	0.0025	0.0108	184.5150	-0.0006	0.0116	157.6426
t_5	0.0033	0.0411	46.0671	0.0054	0.0231	86.4630	0.0049	0.0477	38.3195
t_6	0.0010	0.0115	164.0730	0.0027	0.0112	177.7956	-0.0006	0.0116	157.7187
t_7	0.0018	0.0148	128.1361	0.0008	0.0155	128.7895	-0.0018	0.0142	128.5397
t_8	0.0034	0.0418	45.2414	0.0082	0.0398	50.2067	0.0043	0.0431	42.4350
t_9	0.0113	0.0384	49.2435	0.0164	0.0366	54.5847	0.0119	0.0396	46.1940
t^{NT} **(Proposed)**	**0.0019**	**0.0098**	**193.4924**	**0.0026**	**0.0097**	**206.4190**	**-0.0005**	**0.0096**	**191.3920**

Table 2: Bias, MSEs and PREs of Estimators from the simulated data using $(n = 50, \rho = 0.8)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	-0.0009	0.0191	100.0000	-0.0034	0.0196	100.0000	0.0013	0.0192	100.0000
t_r	0.0066	0.0347	54.9477	0.0043	0.0346	56.6161	0.0041	0.0346	55.5552
t_1	0.0060	0.0313	61.0684	0.0037	0.0311	62.9395	0.0037	0.0311	61.7395
t_2	-0.0004	0.0084	226.1008	-0.0031	0.0107	182.3413	0.0009	0.0144	133.4847
t_3	0.0047	0.0233	81.8023	0.0023	0.0232	84.4916	0.0028	0.0233	82.6662
t_4	0.0020	0.0109	174.9057	-0.0013	0.0080	246.1603	0.0005	0.0073	263.5560
t_5	0.0069	0.0364	52.5000	0.0016	0.0194	100.7946	0.0050	0.0428	44.9477
t_6	0.0009	0.0076	252.1234	-0.0014	0.0078	252.5455	0.0008	0.0076	253.5879
t_7	-0.0008	0.0141	134.9360	-0.0033	0.0145	135.1444	0.0009	0.0143	134.8888
t_8	0.0066	0.0346	55.0957	0.0043	0.0345	56.7685	0.0040	0.0345	55.7049
t_9	0.0159	0.0315	60.5287	0.0137	0.0314	62.4661	0.0135	0.0314	61.2849
t^{NT} **(Proposed)**	**0.0012**	**0.0068**	**280.0115**	**-0.0012**	**0.0068**	**286.3980**	**0.0015**	**0.0069**	**280.6339**

Table 3: Bias, MSEs and PREs of Estimators from the simulated data using $(n = 50, \rho = 0.95)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0008	0.0188	100.0000	0.0001	0.0187	100.0000	-0.0005	0.0185	100.0000
t_r	0.0025	0.0223	84.2682	0.0056	0.0229	82.0133	0.0048	0.0227	81.2666
t_1	0.0021	0.0194	96.8939	0.0052	0.0199	94.1888	0.0043	0.0198	93.3779
t_2	-0.0002	0.0058	326.7297	-0.0001	0.0103	181.1329	-0.0006	0.0144	127.9451
t_3	0.0012	0.0117	160.6394	0.0037	0.0121	155.4624	0.0030	0.0120	154.2707
t_4	0.0001	0.0034	551.1297	0.0007	0.0019	987.9904	-0.0005	0.0057	326.0684
t_5	0.0024	0.0216	86.9547	0.0031	0.0087	214.8687	0.0065	0.0335	55.1073
t_6	-0.0002	0.0019	999.5599	0.0010	0.0019	964.1616	0.0003	0.0019	967.9847
t_7	0.0004	0.0130	145.1986	0.0000	0.0128	145.8917	-0.0006	0.0127	145.4077
t_8	0.0025	0.0223	84.5182	0.0056	0.0228	82.2546	0.0048	0.0227	81.5067
t_9	0.0143	0.0197	95.6826	0.0172	0.0202	92.7057	0.0164	0.0201	91.9944
t^{NT} **(Proposed)**	**0.0007**	**0.0018**	**1019.9059**	**0.0018**	**0.0018**	**1021.9543**	**0.0011**	**0.0019**	**993.3857**

Table 4: Bias, MSEs and PREs of Estimators from the simulated data using $(n = 100, \rho = 0.7)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	-0.0017	0.0098	100.0000	0.0004	0.0087	100.0000	0.0009	0.0094	100.0000
t_r	0.0026	0.0184	53.0841	0.0019	0.0204	42.4586	0.0003	0.0190	49.4758
t_1	0.0023	0.0168	58.3865	0.0017	0.0187	46.5403	0.0002	0.0173	54.3490
t_2	-0.0013	0.0052	187.8762	0.0001	0.0058	150.2079	0.0004	0.0065	144.9367
t_3	0.0016	0.0133	73.5047	0.0013	0.0151	57.3679	0.0000	0.0138	67.9174
t_4	0.0001	0.0066	147.6067	0.0002	0.0052	165.8401	-0.0003	0.0047	197.8829
t_5	0.0025	0.0180	54.4006	0.0010	0.0121	71.6342	0.0002	0.0184	51.0286
t_6	-0.0005	0.0051	192.9463	0.0002	0.0057	153.5763	-0.0003	0.0052	180.5736
t_7	-0.0017	0.0086	114.3190	0.0003	0.0076	114.2743	0.0007	0.0082	114.2805
t_8	0.0026	0.0184	53.1552	0.0019	0.0204	42.5161	0.0003	0.0190	49.5422
t_9	0.0067	0.0168	58.1987	0.0057	0.0187	46.4300	0.0044	0.0173	54.2411
t^{NT} **(Proposed)**	**-0.0005**	**0.0046**	**214.7808**	**0.0005**	**0.0046**	**190.7249**	**0.0002**	**0.0046**	**206.0286**

Table 5: Bias, MSEs and PREs of Estimators from the simulated data using $(n=100, \rho=0.8)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0003	0.0088	100.0000	0.0014	0.0087	100.0000	-0.0001	0.0089	100.0000
t_r	0.0025	0.0163	53.8042	0.0012	0.0154	56.6355	0.0037	0.0164	54.0042
t_1	0.0023	0.0147	59.7075	0.0011	0.0139	62.8414	0.0034	0.0148	59.9022
t_2	0.0001	0.0041	215.0024	0.0010	0.0055	158.0906	0.0000	0.0063	140.6150
t_3	0.0018	0.0110	79.6096	0.0008	0.0104	83.7851	0.0027	0.0111	79.6531
t_4	0.0008	0.0049	178.5665	0.0005	0.0033	266.5558	0.0005	0.0033	265.7992
t_5	0.0025	0.0161	54.4177	0.0007	0.0082	106.5252	0.0044	0.0208	42.5145
t_6	0.0005	0.0036	240.7959	0.0005	0.0036	242.8293	0.0009	0.0037	240.0064
t_7	0.0002	0.0075	116.7247	0.0013	0.0075	116.4761	-0.0001	0.0076	116.5367
t_8	0.0025	0.0163	53.8757	0.0012	0.0154	56.7106	0.0037	0.0164	54.0760
t_9	0.0071	0.0148	59.4707	0.0059	0.0139	62.6298	0.0081	0.0148	59.6255
t^{NT} **(Proposed)**	**0.0007**	**0.0032**	**271.6081**	**0.0009**	**0.0032**	**273.2549**	**0.0010**	**0.0033**	**267.4802**

Table 6: Bias, MSEs and PREs of Estimators from the simulated data using $(n=100, \rho=0.95)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	-0.0010	0.0091	100.0000	-0.0016	0.0090	100.0000	0.0016	0.0091	100.0000
t_r	0.0015	0.0107	84.4449	0.0042	0.0105	85.2456	0.0006	0.0107	85.4748
t_1	0.0013	0.0093	97.1133	0.0039	0.0092	98.0338	0.0005	0.0093	98.3119
t_2	-0.0010	0.0028	319.7014	-0.0011	0.0044	205.5611	0.0010	0.0044	205.7793
t_3	0.0007	0.0056	161.1839	0.0028	0.0055	162.7725	0.0003	0.0056	163.3749
t_4	-0.0003	0.0016	572.7140	0.0004	0.0009	1024.4985	0.0002	0.0009	1023.6567
t_5	0.0015	0.0105	85.9771	0.0025	0.0045	201.4457	0.0006	0.0108	85.0181
t_6	-0.0006	0.0009	1013.6448	0.0005	0.0009	991.1599	0.0002	0.0009	1015.7443
t_7	-0.0010	0.0075	120.6444	-0.0014	0.0074	120.6330	0.0014	0.0076	120.5994
t_8	0.0015	0.0107	84.5702	0.0042	0.0105	85.3720	0.0006	0.0107	85.6015
t_9	0.0072	0.0094	96.4849	0.0097	0.0092	97.0693	0.0064	0.0094	97.7643
t^{NT} **(Proposed)**	**-0.0002**	**0.0009**	**1033.1737**	**0.0008**	**0.0009**	**1034.5000**	**0.0007**	**0.0009**	**1033.6894**

Table 7: Bias, MSEs and PREs of Estimators from the simulated data using $(n=150, \rho=0.7)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	-0.0001	0.0057	100.0000	-0.0003	0.0058	100.0000	-0.0013	0.0059	100.0000
t_r	0.0016	0.0126	44.7587	-0.0001	0.0123	47.2791	0.0031	0.0121	49.1011
t_1	0.0015	0.0115	49.1046	-0.0001	0.0112	51.9211	0.0028	0.0110	53.9295
t_2	-0.0001	0.0031	181.7162	-0.0006	0.0039	150.6136	-0.0010	0.0042	142.5027
t_3	0.0012	0.0093	60.7142	-0.0003	0.0090	64.7545	0.0023	0.0088	67.2468
t_4	0.0004	0.0046	123.1095	-0.0008	0.0031	186.1900	0.0000	0.0030	198.7378
t_5	0.0015	0.0119	47.3749	-0.0005	0.0068	85.1281	0.0032	0.0124	47.8921
t_6	0.0002	0.0034	165.6246	-0.0007	0.0034	171.2259	0.0003	0.0033	179.6694
t_7	-0.0001	0.0052	109.5038	-0.0004	0.0053	109.5958	-0.0013	0.0054	109.4343
t_8	0.0016	0.0126	44.7994	-0.0001	0.0123	47.3215	0.0031	0.0121	49.1452
t_9	0.0040	0.0115	49.0194	0.0025	0.0112	51.8641	0.0055	0.0110	53.7916
t^{NT} **(Proposed)**	**0.0002**	**0.0029**	**196.4188**	**-0.0005**	**0.0028**	**203.8803**	**0.0000**	**0.0029**	**204.5971**

Table 8: Bias, MSEs and PREs of Estimators from the simulated data using $(n=150, \rho=0.8)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0015	0.0057	100.0000	-0.0007	0.0053	100.0000	-0.0008	0.0051	100.0000
t_r	0.0013	0.0104	55.0732	0.0015	0.0103	51.5743	0.0018	0.0106	47.6685
t_1	0.0012	0.0093	61.1281	0.0014	0.0093	57.1714	0.0016	0.0096	52.7831
t_2	0.0010	0.0026	219.0623	-0.0007	0.0032	168.0345	-0.0007	0.0038	134.3207
t_3	0.0011	0.0070	81.5218	0.0010	0.0070	75.8479	0.0012	0.0073	69.3468
t_4	0.0008	0.0032	178.1347	-0.0002	0.0022	239.2738	-0.0004	0.0020	248.9283
t_5	0.0013	0.0098	58.4697	0.0007	0.0055	96.4298	0.0016	0.0095	53.4637
t_6	0.0008	0.0023	246.6781	-0.0001	0.0024	225.6286	-0.0001	0.0023	216.4814
t_7	0.0014	0.0051	110.9712	-0.0007	0.0048	111.0331	-0.0008	0.0046	111.1344
t_8	0.0013	0.0103	55.1221	0.0015	0.0103	51.6198	0.0018	0.0106	47.7109
t_9	0.0042	0.0094	60.9837	0.0043	0.0093	57.0333	0.0044	0.0096	52.6605
t^{NT} **(Proposed)**	**0.0012**	**0.0021**	**274.7876**	**-0.0001**	**0.0020**	**264.3030**	**-0.0001**	**0.0020**	**254.4269**

Table 9: Bias, MSEs and PREs of Estimators from the simulated data using $(n=150, \rho=0.95)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	-0.0005	0.0055	100.0000	0.0003	0.0057	100.0000	0.0002	0.0057	100.0000
t_r	0.0025	0.0068	79.9320	0.0005	0.0068	82.7434	0.0006	0.0070	81.0361
t_1	0.0023	0.0060	91.6983	0.0004	0.0059	95.0496	0.0005	0.0061	93.0535
t_2	-0.0002	0.0016	350.4971	0.0000	0.0027	205.8453	0.0000	0.0037	153.5116
t_3	0.0017	0.0036	150.5198	0.0002	0.0036	157.0349	0.0002	0.0037	153.4825
t_4	0.0009	0.0012	465.8554	-0.0002	0.0006	997.1146	-0.0002	0.0008	683.7876
t_5	0.0024	0.0065	84.7317	0.0001	0.0028	201.6830	0.0010	0.0104	54.8314
t_6	0.0004	0.0006	930.4426	-0.0002	0.0006	951.7586	-0.0002	0.0006	983.6874
t_7	-0.0005	0.0048	113.5067	0.0003	0.0050	113.4466	0.0002	0.0050	113.5356
t_8	0.0025	0.0068	80.0098	0.0005	0.0068	82.8246	0.0006	0.0070	81.1154
t_9	0.0059	0.0060	91.1725	0.0041	0.0060	94.7143	0.0042	0.0062	92.7244
t^{NT} **(Proposed)**	**0.0006**	**0.0006**	**967.0096**	**0.0001**	**0.0006**	**1012.7048**	**0.0001**	**0.0006**	**1016.9664**

Table 10: Bias, MSEs and PREs of Estimators from the simulated data using $(n=200, \rho=0.7)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0003	0.0042	100.0000	0.0011	0.0044	100.0000	0.0006	0.0040	100.0000
t_r	0.0014	0.0086	48.7045	-0.0001	0.0083	52.6990	0.0008	0.0086	46.3144
t_1	0.0013	0.0078	53.4828	-0.0001	0.0075	57.9738	0.0007	0.0079	50.7595
t_2	0.0002	0.0022	188.4708	0.0007	0.0029	150.1463	0.0005	0.0029	139.3176
t_3	0.0011	0.0063	66.6642	-0.0001	0.0060	72.9547	0.0006	0.0064	62.8168
t_4	0.0006	0.0033	127.5792	0.0001	0.0021	206.2146	0.0003	0.0021	187.1618
t_5	0.0014	0.0087	47.7820	-0.0001	0.0046	94.2879	0.0009	0.0098	40.8348
t_6	0.0004	0.0023	178.4897	0.0001	0.0023	190.7068	0.0003	0.0024	165.8004
t_7	0.0003	0.0039	107.0660	0.0010	0.0041	107.1869	0.0006	0.0037	106.9065
t_8	0.0014	0.0086	48.7373	-0.0001	0.0083	52.7345	0.0008	0.0086	46.3451
t_9	0.0032	0.0078	53.4052	0.0019	0.0076	57.9245	0.0026	0.0079	50.7006
t^{NT} **(Proposed)**	**0.0005**	**0.0020**	**203.7741**	**0.0005**	**0.0020**	**216.8944**	**0.0005**	**0.0021**	**193.1211**

Table 11: Bias, MSEs and PREs of Estimators from the simulated data using $(n = 200, \rho = 0.8)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0012	0.0039	100.0000	-0.0003	0.0039	100.0000	-0.0006	0.0040	100.0000
t_r	0.0021	0.0073	53.1070	0.0011	0.0074	52.4634	0.0023	0.0070	57.9187
t_1	0.0020	0.0066	58.9033	0.0010	0.0066	58.1824	0.0021	0.0063	64.3061
t_2	0.0011	0.0018	213.0180	-0.0003	0.0029	133.5086	-0.0005	0.0030	133.7139
t_3	0.0018	0.0049	78.3979	0.0007	0.0050	77.3384	0.0017	0.0047	85.9975
t_4	0.0014	0.0022	176.8055	-0.0002	0.0015	256.5600	0.0000	0.0015	261.1071
t_5	0.0021	0.0073	52.7440	0.0004	0.0032	121.3197	0.0032	0.0109	37.0228
t_6	0.0013	0.0016	237.5390	0.0000	0.0017	230.6345	0.0004	0.0016	253.7401
t_7	0.0012	0.0036	108.1807	-0.0003	0.0036	108.2332	-0.0006	0.0037	108.0248
t_8	0.0021	0.0073	53.1421	0.0011	0.0074	52.4982	0.0023	0.0070	57.9572
t_9	0.0041	0.0066	58.7621	0.0031	0.0066	58.0816	0.0043	0.0063	64.1365
t^{NT} **(Proposed)**	**0.0014**	**0.0014**	**268.2304**	**0.0001**	**0.0014**	**268.4931**	**0.0004**	**0.0015**	**276.7185**

Table 12: Bias, MSEs and PREs of Estimators from the simulated data using $(n = 200, \rho = 0.95)$

	Normal			Skewed (Gamma)			Heavy-tailed (t)		
Estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0007	0.0040	100.0000	-0.0001	0.0041	100.0000	0.0004	0.0041	100.0000
t_r	-0.0005	0.0050	80.7412	0.0010	0.0048	85.3374	0.0003	0.0048	85.0450
t_1	-0.0005	0.0043	92.7100	0.0009	0.0042	98.1532	0.0003	0.0042	97.8092
t_2	0.0001	0.0011	381.2435	-0.0001	0.0024	168.2488	0.0002	0.0022	189.8854
t_3	-0.0005	0.0026	152.8594	0.0006	0.0025	163.0862	0.0001	0.0025	162.4507
t_4	-0.0005	0.0009	443.4488	0.0000	0.0005	893.6424	0.0000	0.0004	1058.3350
t_5	-0.0005	0.0051	79.3473	0.0005	0.0016	251.4768	0.0002	0.0034	121.0382
t_6	-0.0003	0.0004	995.7380	0.0001	0.0004	1012.0788	0.0000	0.0004	1043.1759
t_7	0.0007	0.0037	110.0334	-0.0001	0.0037	109.8956	0.0004	0.0037	109.9229
t_8	-0.0005	0.0050	80.8005	0.0010	0.0048	85.4007	0.0003	0.0048	85.1081
t_9	0.0020	0.0044	92.5789	0.0036	0.0042	97.8224	0.0029	0.0042	97.5633
t^{NT} **(Proposed)**	**-0.0001**	**0.0004**	**1029.6955**	**0.0003**	**0.0004**	**1057.4067**	**0.0002**	**0.0004**	**1066.0789**

3.2 Real Data Application

To further assess the practical applicability and empirical performance of the proposed estimator, three well-established finite population datasets commonly employed in the survey sampling literature were considered.

Table 13: Population descriptive statistics for the real data sets

Parameters	Population I	Population I1	Population III
N	80	40	40
n	20	8	8
$\bar{Y}$	51.8246	50.7858	50.7858
$\bar{X}$	11.2646	2.3033	9.4543
ρ	0.9413	0.8006	0.8349
C_y	0.3542	0.3295	0.3295
C_x	0.7507	0.8406	0.6756
$\beta_1(x)$	1.0500	0.9740	0.8799
$\beta_2(x)$	-0.0634	-0.5344	-0.4622
M_x	7.5750	1.2500	7.0700

Population I: Output (Y) and fixed capital (X) for 80 factories; source: Murthy (1967), as adapted by Zakari *et al.* (2020b). **Population II:** A second subset of the same factory data; source: Murthy (1967). Popu**lation III**: Data from Mukhopadhyay (2009), as adapted by Zakari *et al.* (2020b).

3.2.1 Real Data Results

Table 14: Bias, MSE and PRE of the estimators using Real-life datasets

	Population I			Population II			Population III		
Estimators	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.0000	12.6357	100	0.0000	28.0024	100	0.0000	28.0024	100
t_r	0.6088	18.9793	66.5762	2.4624	95.8641	29.2105	1.3742	49.8536	56.1693
t_1	0.0506	15.2581	82.8131	0.2760	42.0198	66.6410	0.2573	41.0685	68.1846
t_2	0.1134	18.5128	68.2539	3.7353	217.7034	12.8626	0.6588	61.4577	45.5637
t_3	0.0350	14.4503	87.4425	0.3047	43.47728	64.4070	0.2225	39.3031	71.2473
t_4	0.1156	18.6279	67.8321	3.1516	188.0582	14.8903	0.5777	57.3389	48.8367
t_5	0.1170	18.7015	67.5652	0.1896	37.6296	74.4159	0.2131	38.8230	72.1284
t_6	-0.1901	2.7825	454.1132	0.0479	30.4328	92.0139	-0.3213	11.6863	239.6173
t_7	-0.2083	1.8289	690.8907	-0.3242	11.5391	242.6740	-0.3424	10.6117	263.8823
t_8	-0.0372	1.9700	641.1406	-0.0848	12.9992	215.4163	0.1015	9.42415	297.1345
t_9	-0.0355	1.8389	687.1336	-0.2267	11.5152	243.1777	-0.2062	10.4705	267.4409
t^{NT}	-0.1920	1.4400	877.4792	-0.3642	10.0540	278.5200	-0.4265	8.48310	330.0963

3.3 Discussion

The simulation results presented in Tables 1-12 clearly demonstrate the superior finite-sample performance of the proposed estimator (t^{NT}) relative to the conventional sample mean (t_0), the classical ratio estimator (t_r), and the existing modified ratio estimators. Across all combinations of sample size, correlation coefficient, and population distribution, t^{NT} consistently attained the smallest mean square error (MSE) and the largest percentage relative efficiency (PRE). These findings provide empirical support for the theoretical efficiency conditions established in Section 2.3 and confirm that the proposed estimator makes more effective use of the available auxiliary information than the competing estimators.

Tables 1, 4, 7, and 10 correspond to the moderate correlation level ($\rho = 0.70$). At this level of association, the proposed estimator achieved PRE values ranging from 190.72% to 216.89% across all sample sizes and distributions. Although all ratio-type estimators benefited from the auxiliary information, considerable differences in efficiency were observed among them. The classical ratio estimator (t_r) and the estimators t_1, t_5, t_8, and t_9 yielded comparatively low PRE values, in several cases below that of the sample mean estimator. Such behaviour indicates that these estimators were unable to adequately exploit the available auxiliary information under moderate correlation. In contrast, estimators t_4 and t_6 achieved substantial gains in efficiency, with PRE values approaching 198.74% and 192.95%, respectively. Nevertheless, neither estimator matched the efficiency of t^{NT}, which maintained the highest PRE and lowest MSE throughout.

The superiority of the proposed estimator becomes more pronounced as the strength of association increases. From Tables 2, 5, 8, and 11 ($\rho = 0.80$), the PRE of t^{NT} increased to between 254.43% and 286.40%, reflecting the improved utilization of auxiliary information. At this correlation level, estimators t_4 and t_6 emerged as the most competitive alternatives, recording maximum PRE values of 266.56% and 253.74%, respectively. Despite these improvements, t^{NT} consistently maintained a lower MSE than all competing estimators. On the other hand, estimators t_r, t_1, t_5, t_8, and t_8 continued to exhibit relatively low efficiencies, particularly under the heavy-tailed distribution, suggesting greater sensitivity to deviations from ideal distributional assumptions.

The results reported in Tables 3, 6, 9, and 12 ($\rho = 0.95$) provide the strongest evidence of the effectiveness of the proposed estimator. Under this high-correlation scenario, t^{NT} attained PRE values ranging from 967.01% to 1066.08%, indicating an exceptionally large reduction in sampling error relative to the sample mean estimator. Such gains are particularly important in survey practice, where highly correlated auxiliary variables are often deliberately selected to improve estimation precision. Although estimators t_4 and t_6 also recorded substantial efficiency gains at this correlation level, their PRE values remained consistently below those of t^{NT}. For instance, at (n = 200), estimator t_6 achieved PRE values of 995.74%, 1012.08%, and 1043.18% under the Normal, Skewed-Gamma, and Heavy-tailed distributions, respectively, whereas the corresponding PRE values for t^{NT} were 1029.70%, 1057.41%, and 1066.08%. This consistent dominance confirms the ability of the proposed estimator to derive greater precision gains from highly correlated auxiliary variables.

An examination of the MSE values across Tables 1–12 further reveals that all estimators benefited from increases in sample size, which is consistent with the theoretical properties of estimators under SRSWOR. However, the reduction in MSE was most pronounced for t^{NT} . More importantly, the relative efficiency advantage of the proposed estimator was preserved across all sample sizes, demonstrating that its superior performance is not restricted to either small or large samples. The efficiency gains therefore appear to be an inherent characteristic of the estimator structure rather than a consequence of a particular sampling configuration.

The distributional results also highlight the stability of the proposed estimator. Under the Normal, Skewed-Gamma, and Heavy-tailed populations, t^{NT} consistently retained the highest PRE and lowest MSE. In fact, some of its largest efficiency gains were observed under the heavy-tailed distribution, where it attained a maximum PRE of 1066.08% at $\rho = 0.95$ and n = 200. This finding suggests that the estimator remains highly efficient even when the underlying population departs from normality, an important practical consideration since finite populations encountered in real surveys rarely follow ideal distributional forms.

The empirical results presented in Table 14 provide further validation of the simulation findings. Across all three real populations, the proposed estimator, t^{NT} produced the smallest MSE and the largest PRE among all estimators considered. For Population I, t^{NT} achieved a PRE of 877.48%, exceeding the PRE of the most efficient existing estimator, t_7 , which recorded 690.89%. Similar result was observed for Population II, where t^{NT} attained a PRE of 278.52% compared with 243.18% for the t_9 estimator, and for Population III, where it recorded a PRE of 330.10% relative to 297.13% for t_8 . These results confirm that the efficiency gains observed in the simulation study are not merely theoretical but are realized in practical finite population settings.

4. Conclusion

This paper has proposed an improved power-transformed ratio estimator of the finite population mean under SRSWOR. By incorporating the known population correlation coefficient and coefficient of variation of the auxiliary variable, together with an optimally determined unknown weight, the estimator achieves the minimum MSE of the linear regression estimator, the theoretical efficiency lower bound for ratio-type estimators. Closed-form expressions for bias and minimum MSE were derived; efficiency conditions establish that the proposed estimator dominates the sample mean, the classical ratio estimator, and other modified ratio estimators from the literature.

These theoretical advantages were confirmed by a Monte Carlo simulation and three real datasets. The proposed estimator is therefore recommended for practical survey applications, particularly in agriculture, health economics, and social sciences, where a single positively correlated auxiliary variable is available and sampling is conducted without replacement.

Future work may extend the proposed framework to stratified, cluster, or two-phase sampling designs, and to settings with two or more auxiliary variables, where further efficiency gains are anticipated.

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References

Cochran, W. G. (1940). The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce. *Journal of Agricultural Science*, 30(2), 262–275.

Ekpenyong, E. J., Okoro, L. C. & Nelson, T. O. (2025). A generalized shrinkage-type estimator of population mean in simple random sampling under conventional and non-conventional measures of auxiliary variables. *Iraqi Statisticians Journal*, 2(2), 33–48.

Jerajuddin, M. & Kishun, J. (2016). Modified ratio estimators for population mean using size of the sample selected from the population. *International Journal for Research in Applied Science and Engineering Technology*, 4(2), 10–16.

Kadilar, C. & Cingi, H. (2004). Ratio estimators in simple random sampling. *Applied Mathematics and Computation*, 151, 893–902.

Khoshnevisan, M., Singh, R., Chauhan, P., Sawan, N., & Smarandache, F. (2007). A general family of estimators for estimating population mean using known value of some population parameter(s). *Far East Journal of Theoretical Statistics*, *22*(2), 181–191.

Muhammad, I., Zakari, Y. & Audu, A. (2023). Enhanced ratio-type estimator for finite population mean using auxiliary variable in simple random sampling. *NIPES Journal of Science and Technology Research*, 5(1), 242–252.

Mukhopadhyay, P. (2009). *Theory and Methods of Survey Sampling* (2nd edn). PHI Learning, New Delhi.

Murthy, M. N. (1964). Product method of estimation. Sankhyā: *The Indian Journal of Statistics, Series* A, 26, 69–74.

Singh, H.P., & Horn, S. (1998). An alternative estimator for multi-character surveys, *Metrika*, 48, 99-107

Singh, H. P. & Tailor, R. (2003). Use of known correlation coefficient in estimating the finite population mean. *Statistics in Transition*, 6(4), 555–560.

Singh, H. P., Tailor, R. & Kakran, M. S. (2004). An improved estimation of population mean using power transformation. *Journal of the Indian Society of Agricultural Statistics*, 58(2), 223–230.

Sisodia, B. V. S. & Dwivedi, V. K. (1981). A modified ratio estimator using coefficient of variation of auxiliary variable. *Journal of the Indian Society of Agricultural Statistics*, 33(1), 13–18.

Subramani, J. & Kumarpandiyan, G. (2013). New modified ratio estimation using the median of auxiliary variable. *Pakistan Journal of Statistics and Operations Research*, 9(2), 137–145.

Upadhyaya, L. N. & Singh, H. P. (1999). Use of transformed auxiliary variable in estimating the finite population mean. *Biometrical Journal*, 41(5), 627–636.

Yadav, S. K., Diksha, A., Tuba, K. & Zaman, T. (2024). An efficient family of ratio type estimators for simple random sampling. *Journal of Science and Arts*, 24(1), 69–94.

Yadav, S. K., Dixit, M. K., Dungana, H. N. & Mishra, S. S. (2019). Improved estimators for estimating average yield using auxiliary variable. *International Journal of Mathematical, Engineering and Management Sciences*, 4(5), 1228–1238.

Yan, Z. & Tian, B. (2010). Ratio method to the mean estimation using coefficient of skewness of auxiliary variable. ICICA, Part II, CCIS, 106, 103–110.

Zakari, Y., Muili, J. O., Tela, M. N., Danchadi, N. S. & Audu, A. (2020). Use of unknown weight to enhance ratio-type estimator in simple random sampling. *Lapai Journal of Applied and Natural Sciences*, 5(1), 74–81.

Zaman, Q., Bushra, H., Qasim, S., Anwar, M., Kousar, S., Ali, F. & Ayub, G. (2024). A novel class of ratio estimators incorporating auxiliary variables in simple random sampling for estimating population mean. *Power System Technology*, 48(3), 609–621.